

## CONNECTEDNESS IN THE PLURI-FINE TOPOLOGY

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**ABSTRACT.** We study connectedness in the pluri-fine topology on  $\mathbb{C}^n$  and obtain the following results. If  $\Omega$  is a pluri-finely open and pluri-finely connected set in  $\mathbb{C}^n$  and  $E \subset \mathbb{C}^n$  is pluripolar, then  $\Omega \setminus E$  is pluri-finely connected. The proof hinges on precise information about the structure of open sets in the pluri-fine topology: Let  $\Omega$  be a pluri-finely open subset of  $\mathbb{C}^n$ . If  $z$  is any point in  $\Omega$ , and  $L$  is a complex line passing through  $z$ , then obviously  $\Omega \cap L$  is a finely open neighborhood of  $z$  in  $L$ . Now let  $C_L$  denote the finely connected component of  $z$  in  $\Omega \cap L$ . Then  $\cup_{L \ni z} C_L$  is a pluri-finely connected neighborhood of  $z$ . As a consequence we find that if  $v$  is a finely plurisubharmonic function defined on a pluri-finely connected pluri-finely open set, then  $v = -\infty$  on a pluri-finely open subset implies  $v \equiv -\infty$ .

**Key words:** Fine topology, Subharmonic functions, Harmonic measure, Pluri-fine topology, Plurisubharmonic functions, Pluripolar sets, Pluripolar hulls.

## 1. INTRODUCTION

The pluri-fine topology on an open set  $\Omega$  in  $\mathbb{C}^n$  is the coarsest topology on  $\Omega$  that makes all plurisubharmonic (PSH) functions on  $\Omega$  continuous. Understanding the pluri-fine topology is a first step towards understanding pluri-fine potential theory and pluri-fine holomorphy. There is now some evidence, see [7, 8], that a good theory of finely plurisubharmonic and finely holomorphic functions may be needed for dealing with questions about pluripolarity.

In order to avoid cumbersome expressions like “locally pluri-finely connected sets”, we adopt the following **convention**: Topological notions referring to the pluri-fine topology will be qualified by the prefix  $\mathcal{F}$  to distinguish them from those pertaining to Euclidean topology. For example,  $\mathcal{F}$ -open,  $\mathcal{F}$ -domain (it means  $\mathcal{F}$ -open and  $\mathcal{F}$ -connected),  $\mathcal{F}$ -component,... In view of the fact that the pluri-fine topology restricted to a complex line coincides with the fine topology on that line, this convention can be used in the one

dimensional setting, where we will work with the fine topology, at the same time.

In a previous paper [10] we proved that the pluri-fine topology is locally connected, and we stated the following theorem.

**Theorem 1.1.** *Let  $U$  be an  $\mathcal{F}$ -domain in  $\subseteq \mathbb{C}^n$ . If  $E$  is a pluripolar set, then  $U \setminus E$  is  $\mathcal{F}$ -connected.*

We referred to the corresponding result in fine potential theory for a proof, but this is unjustified, as Norman Levenberg [20] noticed. Here we will give a proof of Theorem 1.1. It will be a consequence of technical result, Proposition 4.1. A slightly weaker but easy formulation is as follows. For a point  $z$  in an  $\mathcal{F}$ -open subset  $\Omega \subset \mathbb{C}^n$  and  $L$  a complex line passing through  $z$ , denote by  $C_L$  the  $\mathcal{F}$ -component of  $z$  in the  $\mathcal{F}$ -open set  $\Omega \cap L$ .

**Theorem 1.2.** *Let  $\Omega$  be an  $\mathcal{F}$ -open subset of  $\mathbb{C}^n$  and let  $z \in \Omega$ . Then  $\cup_{L \ni z} C_L$  is an  $\mathcal{F}$ -neighborhood of  $z$  which is  $\mathcal{F}$ -connected.*

Note that  $C_L$  is  $\mathcal{F}$ -open in  $L$ , because the fine topology is locally connected, cf. [11]. We will also present here (cf. Corollary 3.3) a different proof of this fact. Theorem 1.2 includes the main result in [10]:

**Corollary 1.3.** *The pluri-fine topology on an open set  $\Omega$  in  $\mathbb{C}^n$  is locally connected.*

The proof of the local connectedness in the present paper is conceptually easier, but uses much more information on the structure of  $\mathcal{F}$ -open sets, whereas the proof in [10] reveals that one can find an explicit  $\mathcal{F}$ -neighborhood basis consisting of  $\mathcal{F}$ -domains, cf. Remark 2.4 below.

In Section 5 we give a definition of  $\mathcal{F}$ -plurisubharmonic functions and obtain the following result.

**Theorem 1.4.** *Let  $f$  be an  $\mathcal{F}$ -plurisubharmonic function on an  $\mathcal{F}$ -domain  $\Omega$ . If  $f = -\infty$  on an  $\mathcal{F}$ -open subset of  $\Omega$ , then  $f \equiv -\infty$ .*

In fine potential theory a much more precise result holds (cf. [12], Theorem 12.9). Nevertheless, Theorem 1.4 will turn out to be very useful. Besides its key role in the proof of Theorem 1.1, it entails an interesting consequence for the study of pluripolar hulls of graphs. Namely, the main result of Edlund and Jöricke in [8] can be extended to functions of several complex variables. We will discuss this in Section 6.

## 2. PRELIMINARIES

We fix the following notation:  $\mathbb{D}(a, r) = \{|z - a| < r\}$ ,  $\mathbb{D} = \mathbb{D}(0, 1)$ ,  $C(a, r) = \{|z - a| = r\}$ , while  $B(a, R) = \{\|z - a\| < R\} \subset \mathbb{C}^n$ .

**2.1. Harmonic measure.** Let  $\Omega$  be an open set in the complex plane  $\mathbb{C}$  and let  $E \subseteq \overline{\Omega}$ . Subharmonic functions on  $\Omega$  are denoted by  $\text{SH}(\Omega)$ , while  $\text{SH}^-(\Omega) = \{u \in \text{SH}(\Omega) : u \leq 0\}$ . The harmonic measure (or the relative extremal function) of  $E$  (relative to  $\Omega$ ) at  $z \in \Omega$  is defined as follows (see, e.g. [22])

$$\omega(z, E, \Omega) = \sup\{u(z) : u \in \text{SH}_-(\Omega), \limsup_{\Omega \ni v \rightarrow \zeta} u(v) \leq -1 \text{ for } \zeta \in E\}.$$

This function need not be subharmonic in  $\Omega$ , but its upper semi-continuous regularization

$$\omega(z, E, \Omega)^* = \limsup_{\Omega \ni v \rightarrow z} \omega(v, E, \Omega) \geq \omega(v, E, \Omega)$$

is subharmonic. If  $E$  is a closed subset of  $\Omega$ , then  $\omega(\cdot, E, \Omega)$  coincides with the Perron solution of the Dirichlet problem in  $\Omega \setminus E$  with boundary values  $-1$  on  $\partial E \cap \Omega$  and  $0$  on  $\partial\Omega \setminus \partial E$ .

Recall the following result, cf. [5] and [6].

**Theorem 2.1.** *Let  $\Omega$  be a bounded open subset of  $\mathbb{C}$ . If  $E \subset \Omega$  is a Borel set, then there exists an increasing sequence of compact sets  $K_j \subset E$  such that  $\omega(z, K_j, \Omega)^* \downarrow \omega(z, E, \Omega)^*$ .*

Let  $E \subset \mathbb{D}$ . We associate to  $E$  its circular projection

$$E^\circ = \{|z| : z \in E\}.$$

There is extensive literature on harmonic measure and its behavior under geometric transformations such as projection, symmetrization, and polarization. We refer to [22] and the survey article [4] and the references therein.

Our main tool in this paper is the following classical theorem of A. Beurling and R. Nevanlinna related to the Carleman-Milloux problem, cf. [3] and [21]. See also [4].

**Theorem 2.2.** *Let  $F \subset \mathbb{D}$  be compact. Let  $F^\circ$  be its circular projection. Then*

$$\omega(z, F, \mathbb{D}) \leq \omega(-|z|, F^\circ, \mathbb{D}), \text{ for all } z \in \mathbb{D} \setminus F.$$

We will need the following result, observed in [2], cf. [10], Lemma 3.1.

**Theorem 2.3.** *Sets of the form*

$$\Omega_{B(z,r),\varphi,c} = \{w \in B(z,r) : \varphi(w) > c\},$$

where  $\varphi \in \text{PSH}(B(z,r))$  and  $c \in \mathbb{R}$ , constitute a base of the pluri-fine topology on  $\mathbb{C}^n$ .

*Remark 2.4.* It follows from the proof of Theorem 1.1 in [10] that each point in an  $\mathcal{F}$ -open set has a neighborhood basis consisting of  $\mathcal{F}$ -domains of the form  $\Omega_{B(z,r),\varphi,c}$ .

## 3. ESTIMATES FOR SUBHARMONIC FUNCTIONS

**Lemma 3.1.** *For every  $d < c < 0$  there exists  $\kappa > 0$  such that for every  $\varphi \in \text{SH}_-(\mathbb{D})$  with  $\varphi(0) > c$  and for every point  $a$  in the  $\mathcal{F}$ -open set*

$$V = \{z \in \mathbb{D}(0, 1/8) : \varphi(z) > c\}$$

*the set*

$$\Omega = \{z \in \mathbb{D} : \varphi(z) \geq d\}$$

*contains a circle  $C(a, \delta_{\varphi,a})$  with radius  $\delta_{\varphi,a} > \kappa$ .*

*Proof.* After multiplying  $\varphi$  by a constant we can assume that  $d = -1$ . Moreover, we may assume that the set  $E = \{z \in \mathbb{D} : \varphi(z) < d\}$  is non empty since otherwise the lemma trivially holds.

Let  $a \in V$  be fixed. We will first prove the following estimate

$$(3.1) \quad \varphi(a) \leq \omega(a, E_a^\circ, \mathbb{D}(a, 3/4))^*,$$

where  $E_a^\circ = \{a + |z - a| : z \in E \cap \mathbb{D}(a, 3/4)\}$ .

Let  $f$  be the function  $f(z) = z + a$ . Note that the circular projection commutes with  $f^{-1}$ , i.e.,  $f^{-1}(E_a^\circ) = (f^{-1}(E \cap \mathbb{D}(a, 3/4)))^\circ$ . Hence, to prove (3.1) it is enough, in view of the conformal invariance of the harmonic measure, to prove that the estimate (3.1) holds for the particular point  $a = 0$ , i.e.,

$$(3.2) \quad \varphi(0) \leq \omega(0, E^\circ, \mathbb{D}(0, 3/4))^*.$$

By Theorem 2.1, there is an increasing sequence of compact subset  $K_j$  of  $E^\circ$  such that

$$(3.3) \quad \omega(0, K_j, \mathbb{D}(0, 3/4))^* = \omega(0, K_j, \mathbb{D}(0, 3/4)) \downarrow \omega(0, E^\circ, \mathbb{D}(0, 3/4))^*.$$

The equality in (3.3) holds because  $K_j$  is compact and  $0 \notin K_j$ . Let  $\varepsilon > 0$ . It follows from (3.3) that there exists a natural number  $j_0$  such that

$$(3.4) \quad \omega(0, K_{j_0}, \mathbb{D}(0, 3/4)) \leq \omega(0, E^\circ, \mathbb{D}(0, 3/4))^* + \varepsilon.$$

We can find a compact set  $L \subset E$  such that  $L^\circ = K_{j_0}$ . By Theorem 2.2 together with inequality (3.4) we get

$$(3.5) \quad \omega(0, L, \mathbb{D}(0, 3/4)) \leq \omega(0, K_{j_0}, \mathbb{D}(0, 3/4)) \leq \omega(0, E^\circ, \mathbb{D}(0, 3/4))^* + \varepsilon.$$

Since  $L \subset E$ , and  $\varphi(z) < -1$ , for all  $z \in E$ , inequality (3.5) implies the following estimate

$$(3.6) \quad \varphi(0) \leq \omega(0, E^\circ, \mathbb{D}(0, 3/4))^* + \varepsilon.$$

As  $\varepsilon$  is arbitrary, the estimate (3.2), and therefore also (3.1), follows.

Let now  $\alpha \in ]0, 1/4[$  be a constant such that

$$I = \{z \in \mathbb{D}(a, 3/4) : \Im z = \Im a, \text{ and } \Re a + \alpha \leq \Re z \leq 1/2\} \subset E_a^\circ.$$

Then by (3.1)

$$(3.7) \quad \varphi(a) \leq \omega(a, E_a^\circ, \mathbb{D}(a, 3/4))^* \leq \omega(a, I, \mathbb{D}(a, 3/4)).$$

Again by the conformal invariance, (3.7) yields

$$(3.8) \quad \varphi(a) \leq \omega(0, f^{-1}(I), \mathbb{D}(0, 3/4)).$$

Since  $f^{-1}(I) = [\alpha, 1/2 - \Re a]$ , it follows that

$$(3.9) \quad \varphi(a) \leq \omega(0, [\alpha, 3/8], \mathbb{D}(0, 3/4)).$$

Let  $\alpha_j \downarrow 0$  be a sequence decreasing to 0. Since  $\alpha_j \rightarrow \omega(0, [\alpha_j, 3/8], \mathbb{D}(0, 3/4))$  decreases to  $-1$  (see e.g [17], Theorem 8.38), there exists a constant  $0 < \kappa < 3/8$  depending only on  $c$  but not on the function  $\varphi$  such that

$$(3.10) \quad \omega(0, [\kappa, 3/8], \mathbb{D}(0, 3/4)) < c.$$

The last inequality together with (3.9) hence show that for all  $a \in V$ , the interval  $I = \{z \in \mathbb{D}(a, 3/4) : \Im z = \Im a, \text{ and } \kappa + \Re a \leq \Re z \leq 3/8\}$  can not be a subset of  $E_a^\circ$ . We conclude that there exists a  $\delta_{\varphi, a} \in [\kappa, 1/2]$  such that

$$(3.11) \quad \{z : |z - a| = \delta_{\varphi, a}\} \subset \Omega = \{z \in \mathbb{D} : \varphi(z) \geq d\}.$$

□

For our purposes we don't need precise estimates for  $\kappa$ , but these can be easily obtained using the formula of the harmonic measure of an interval, cf. [1].

**Lemma 3.2.** *Let  $\varphi \in \text{SH}(\mathbb{D})$  such that  $0 \leq \varphi \leq 1$ . Let  $U$  be the  $\mathcal{F}$ -open subset of  $\mathbb{D}$  where  $\varphi > 0$ . Suppose that there exists a piecewise- $C^1$  Jordan curve  $\gamma \subset \mathbb{D}$  such that  $\gamma \subset U$ . Let  $\Gamma$  be the bounded component of the complement of  $\gamma$ . Then  $W = U \cap \Gamma$  is polygonally connected and, hence  $\mathcal{F}$ -connected.*

*Proof.* We follow Fuglede's ideas in [13], section 5. A square shall be an open square with sides parallel to the coordinate axes. The square centered at  $z$  with diameter  $d$  will be denoted by  $Q(z, d)$ , its boundary by  $S(z, d)$ .

Let  $n \geq 1$  be a natural number. For every  $z \in \gamma$  there exists  $0 < d_z < 1/n$  such that  $\varphi > \varphi(z)/2$  on  $S(z, d_z) \subset \mathbb{D}$ . This may be proved similarly as the corresponding well-known statement for circles, cf. [17], Theorem 10.14. The squares  $Q(z, d_z)$  cover  $\gamma$ . By compactness we can select a finite subcover  $\{Q(z_j, d_j), j = 1 \dots m_n\}$ , that is minimal in the sense that no square can be removed without loosing the covering property. Now  $\Omega_n = \cup_{j=1}^{m_n} Q(z_j, d_j)$  is an open neighborhood of  $\gamma$ , the boundary of which is contained in

$$\{\varphi > \frac{1}{2} \min_{1 \leq j \leq m_n} \{\varphi(z_j)\}\}.$$

Since  $\gamma$  is locally connected, the boundary of  $\Omega_n$  will consist of two polygonal curves if  $n$  is sufficiently large. One of these components, say,  $\gamma_n$  is contained in  $\Gamma$ . Denote by  $\Gamma_n$  the bounded component of the complement of  $\gamma_n$ .

Let  $0 < \varepsilon < \frac{1}{2} \min_{1 \leq j \leq m_n} \{\varphi(z_j)\}$  and  $K_\varepsilon^n = \{\varphi \geq \varepsilon\} \cap \bar{\Gamma}_n$ . Then  $K_\varepsilon^n$  is a compact subset of  $\mathbb{D}$ . Since  $\partial\Gamma_n (= \gamma_n)$  is contained in  $K_\varepsilon^n$ , an easy application of the maximum principle shows that  $K_\varepsilon^n$  is connected. Let  $z_1, z_2$  be points in  $K_\varepsilon^n$ . Repeating the above argument, we find for every  $\delta > 0$  a polygonal curve  $C$  contained in  $K_{\varepsilon/2}^n$ , such that  $d(z_1, C), d(z_2, C) < \delta$ . A well known result, cf. [22] states that for  $z \in \mathbb{D}$  and almost all  $\theta \in [0, 2\pi]$

$$\lim_{r \rightarrow 0} \varphi(z + re^{i\theta}) = \varphi(z).$$

The conclusion is that there exists a polygonal line in  $K_{\varepsilon/2}^n$  that connects  $z_1$  with  $z_2$ . Letting  $\varepsilon \rightarrow 0$  we conclude that  $U \cap \bar{\Gamma}_n = \cup_{\varepsilon > 0} K_\varepsilon^n$  is polygonally connected. Since every interval is  $\mathcal{F}$ -connected, cf. [11], so is  $U \cap \bar{\Gamma}_n$ . Finally, since  $W = \bigcup_{n \geq 1} U \cap \bar{\Gamma}_n$ , we conclude that  $W$  is  $\mathcal{F}$ -connected.  $\square$

As an easy consequence of Lemma 3.2 we have the following

**Corollary 3.3.** *The fine topology on  $\mathbb{C}$  is locally connected.*

*Proof.* Let  $z \in \mathbb{C}$  and let  $U \subseteq \mathbb{C}$  be an  $\mathcal{F}$ -neighborhood of  $z$ . By Theorem 2.3 there exists an  $\mathcal{F}$ -open  $\mathcal{F}$ -neighborhood  $V = \Omega_{\mathbb{D}(z,r),\varphi,c} \subseteq U$  of  $z$ . Without loss of generality we may assume that

$$V = \mathbb{D}(z, 1) \cap \{\varphi > 0\},$$

as noted before  $V$  contains arbitrarily small circles about  $z$ . Let  $\partial\mathbb{D}(z, r)$  be one of them. Then by Lemma 3.2,  $V \cap \mathbb{D}(z, r)$  is an  $\mathcal{F}$ -neighborhood of  $z$  which is an  $\mathcal{F}$ -domain.  $\square$

*Remark 3.4.* Besides the elementary proof that we presented here there are at least three proofs of this corollary. The first one was found by Fuglede [11], who gave a second proof in [12], page 92. Fuglede [15] observed, furthermore, that since our proof of the local connectedness in [10] does not use the fact that the fine topology on  $\mathbb{C} = \mathbb{R}^2$  is locally connected, it provides of course (for  $n = 1$ ) a third proof of that fact.

#### 4. STRUCTURE OF $\mathcal{F}$ -OPEN SETS

We start this Section with the technical result that was alluded to in the introduction.

**Proposition 4.1.** *Let  $U \subseteq \mathbb{C}^n$  be an  $\mathcal{F}$ -open subset and let  $a \in U$ . Then there exists a constant  $\kappa = \kappa(U, a)$  and an  $\mathcal{F}$ -neighborhood  $V \subset U$  of  $a$  with the property that for any complex line  $L$  through  $v \in V$  the  $\mathcal{F}$ -component of the  $\mathcal{F}$ -open set  $U \cap L$  that contains  $v$ , contains a circle about  $v$  with radius at least  $\kappa$ .*

*Proof.* Let  $a \in U$ . By Theorem 2.3 there exist two constants  $r > 0$ ,  $d < 0$  and a plurisubharmonic function  $\varphi \in \text{PSH}(B(a, r))$  such that  $\Omega = \{z \in B(a, r) : \varphi(z) \geq d\}$  is an  $\mathcal{F}$ -neighborhood of  $a$  contained in  $U$ . Since the pluri-fine topology is biholomorphically invariant, there is no loss of generality if we assume that  $r = 2$ ,  $\varphi \leq 0$ ,  $a = 0$  and  $\varphi(0) = d/2$ . Let

$$V = \{z \in B(0, 1/8) : \varphi(z) > d/2\}.$$

Let  $v \in V$  and let  $L$  be a complex line through  $v$ . The restriction  $\varphi_L$  of  $\varphi$  to  $B(v, 1) \cap L$  is subharmonic and satisfies the conditions of Lemma 3.1. Consequently, there exists a constant  $\kappa$  depending only on  $d$ , but not on  $\varphi_L$ , such that the set  $\{z \in B(v, 1) : \varphi(z) \geq d\} \cap L$ , and therefore  $U \cap L$ , contains a circle with radius  $\delta_{\varphi_L, v} \in (\kappa, \frac{1}{2})$  about  $v$ . It follows from Lemma 3.2 that the set  $U \cap L \cap B(v, \delta_L)$  is  $\mathcal{F}$ -connected. This completes the proof of the proposition.  $\square$

*Proof of Theorem 1.2.* Let  $V \subseteq \Omega$  be an  $\mathcal{F}$ -neighborhood of  $z$  provided by Theorem 2.3. Without loss of generality we may assume

$$V = B(z, 1) \cap \{\varphi > 0\},$$

for some  $\varphi \in \text{PSH}(B(z, 1))$ . Recall that for a complex line  $L$  through  $z$ ,  $C_L$  is the  $\mathcal{F}$ -component of  $\Omega \cap L$  that contains  $z$ . It is immediate that  $\cup_{z \in L} C_L$  is  $\mathcal{F}$ -connected. We denote by  $\tilde{C}_L$  the  $\mathcal{F}$ -component of  $V \cap L$  that contains  $z$ . By Lemma 3.1 together with Lemma 3.2 we can find a constant  $\kappa > 0$  such that  $V \cap B(z, \kappa) \cap L \subseteq \tilde{C}_L$ . As  $\tilde{C}_L$  is clearly contained in  $C_L$ ,  $V \cap B(z, \kappa)$  is a subset of  $\cup_{L \ni z} C_L$ . This proves that  $\cup_{L \ni z} C_L$  is an  $\mathcal{F}$ -neighborhood of  $z$ .  $\square$

The next gluing lemma was used in [10]. The lemma is actually an immediate consequence of Fuglede's results, but it seemed interesting to find a proof that avoids the use of the fine potential theory machinery. Here we give a short direct proof.

**Lemma 4.2.** *Let  $v \in \text{SH}(D)$  for some domain  $D \subset \mathbb{C}$ . Suppose that  $v \geq 0$  and that there exist nonempty, disjoint  $\mathcal{F}$ -open sets  $D_1, D_2 \subset D$  such that*

$$\{v > 0\} = D_1 \cup D_2.$$

*Then the function  $v_1$  defined by*

$$(4.1) \quad v_1(z) = \begin{cases} 0 & \text{if } z \in D \setminus D_1, \\ v(z) & \text{if } z \in D_1, \end{cases}$$

*is subharmonic in  $D$ .*

*Proof.* For  $\varepsilon > 0$  let  $D_i(\varepsilon) = D_i \cap \{v \geq \varepsilon\}$ , ( $i = 1, 2$ ). We claim that  $D_1(\varepsilon)$  is closed in  $D$ . Indeed, take a sequence  $\{x_n\}$  in  $D_1(\varepsilon)$  that converges to

$y \in D$ . Since  $\{v \geq \varepsilon\}$  is closed in  $D$ ,  $v(y) \geq \varepsilon$ . Thus  $y \in D_1 \cup D_2$ . Suppose that  $y \in D_2$ . Again there exists  $r > 0$  such that  $C(y, r)$  is contained in the  $\mathcal{F}$ -open set  $\{v > \varepsilon/2\}$ . By Lemma 3.2 the set

$$U = \mathbb{D}(y, r) \cap \{v > \varepsilon/2\}$$

is an  $\mathcal{F}$ -connected subset of  $D_1 \cup D_2$ . Since  $U \cap D_2$  is non-empty,  $U \cap D_1 = \emptyset$ . This contradicts the fact that  $U$  contains  $x_n$  for  $n$  sufficiently large. Thus  $y \in D_1$  and hence  $D_1(\varepsilon)$ , which proves the claim.

Now define

$$(4.2) \quad v_\varepsilon(z) = \begin{cases} \varepsilon & \text{if } z \in D \setminus D_1(\varepsilon), \\ v(z) & \text{if } z \in D_1(\varepsilon). \end{cases}$$

The function  $v_\varepsilon$  is clearly upper semicontinuous in  $D$  and it satisfies the mean value inequality in  $D \setminus D_1(\varepsilon)$ . Let  $a \in D_1(\varepsilon)$  and denote by  $\bar{v}(a, r)$  the mean value of  $v$  over the circle  $C(a, r)$ . Since  $D_2(\varepsilon)$  is similarly closed,  $v \leq v_\varepsilon$  on  $C(a, r)$  for sufficiently small  $r$ . Consequently,

$$v_\varepsilon(a) = v(a) \leq \bar{v}(a, r) \leq \bar{v}_\varepsilon(a, r).$$

This proves that  $v_\varepsilon$  is subharmonic in  $D$ . Finally, the sequence  $\{v_{1/n}\}_n$  decreases to  $v_1$ , showing that  $v_1$  is subharmonic.  $\square$

It was proved by Gamelin and Lyons in [16] that an  $\mathcal{F}$ -open subset of  $\mathbb{C}$  is  $\mathcal{F}$ -connected if and only if it is connected with respect to the usual topology on  $\mathbb{C}$ . The next example shows that this result has no analog in  $\mathbb{C}^n$  for  $n > 1$ .

**Example 4.3.** There exists an  $\mathcal{F}$ -open set  $U \subset \mathbb{C}^2$ , which is connected but not  $\mathcal{F}$ -connected. Indeed, consider the set

$$\Gamma = \{(x, y) \in \mathbb{C}^2 : y = e^{1/x}, -1 \leq x < 0\}.$$

As was proved by the second author, cf. [23], one can find a plurisubharmonic function  $\varphi \in \text{PSH}(B(0, 2))$  such that  $\varphi|_\Gamma = -\infty$  and  $\varphi(0) = 0$ . Let  $V = \{\varphi > -1/2\}$  and  $W = \{\varphi < -1/2\}$ . Let  $W_1$  be the connected component of  $W$  that contains  $\Gamma$ , and let  $V_1$  the  $\mathcal{F}$ -component of  $V$  that contains 0. Since the pluri-fine topology is locally connected,  $V_1$  is  $\mathcal{F}$ -open. Of course  $V_1$  is connected since it is already  $\mathcal{F}$ -connected. Observe now that  $U = V_1 \cup W_1$  is  $\mathcal{F}$ -open and  $\mathcal{F}$ -disconnected. On the other hand, since  $W_1 \cup \{0\}$  is clearly connected, the  $\mathcal{F}$ -open set  $U = V_1 \cup W_1 = V_1 \cup (W_1 \cup \{0\})$  is connected.

## 5. FINELY PLURISUBHARMONIC FUNCTIONS

As far as we know there is no generally accepted definition of  $\mathcal{F}$ -pluri-subharmonic functions. The following, cf. [9], seems quite natural.



**Definition 5.1.** A function  $f : \Omega \longrightarrow [-\infty, \infty[$  ( $\Omega$  is  $\mathcal{F}$ -open in  $\mathbb{C}^n$ ) is called  $\mathcal{F}$ -plurisubharmonic if  $f$  is  $\mathcal{F}$ -upper semicontinuous on  $\Omega$  and if the restriction of  $f$  to any complex line  $L$  is finely subharmonic or  $\equiv -\infty$  on any  $\mathcal{F}$ -component of  $\Omega \cap L$ .

It follows immediately from this definition and general properties of finely subharmonic functions that any usual plurisubharmonic function is  $\mathcal{F}$ -plurisubharmonic where it is defined. Moreover,  $\mathcal{F}$ -plurisubharmonic functions in an  $\mathcal{F}$ -open set  $\Omega$  form a convex cone, which is stable under pointwise infimum for lower directed families, under pointwise supremum for finite families, and closed under pluri-finely locally uniform convergence. See [12], page 84-85, and [9].

Clearly, an  $\mathcal{F}$ -plurisubharmonic function  $f$  on an  $\mathcal{F}$ -open set  $\Omega$  has an  $\mathcal{F}$ -plurisubharmonic restriction to every  $\mathcal{F}$ -open subset of  $\Omega$ . Conversely, suppose that  $f$  is  $\mathcal{F}$ -plurisubharmonic in some  $\mathcal{F}$ -neighborhood of each point of  $\Omega$ . Then  $f$  is  $\mathcal{F}$ -plurisubharmonic in  $\Omega$ , see. [12], page 70. We shall refer to this by saying that the  $\mathcal{F}$ -plurisubharmonic functions have the *sheaf property*.

**Theorem 5.2.** *Let  $f$  be an  $\mathcal{F}$ -plurisubharmonic function on a  $\mathcal{F}$ -domain  $\Omega$ . If  $f = -\infty$  on an  $\mathcal{F}$ -open subset  $U$  of  $\Omega$ , then  $f \equiv -\infty$ .*

*Proof.* Without loss of generality we can assume that  $U$  is the  $\mathcal{F}$ -interior of the set  $\{f = -\infty\}$ . Let  $z_0 \in \Omega$  be an  $\mathcal{F}$ -boundary point of  $U$ . After scaling we can assume that  $z_0 = 0$  and that

$$(5.1) \quad V = B(0, 1) \cap \{\varphi > 0\} \subset \Omega.$$

is an  $\mathcal{F}$ -neighborhood of 0 defined by a PSH-function  $\varphi$  on  $B(0, 1)$  with  $\varphi(0) = 1$ . Then

$$(5.2) \quad V_{1/2} = B(0, 1/2) \cap \{\varphi > 1/2\}$$

is a smaller  $\mathcal{F}$ -neighborhood of 0. For every  $z \in V_{1/2} \cap U$  the function  $\varphi$  is defined on  $B(z, 1/2)$  and  $B(z, 1/2) \cap \{\varphi > 0\}$  is an  $\mathcal{F}$ -neighborhood of  $z$  contained in  $\Omega$ .

By Lemma 3.1 together with Lemma 3.2 there exists  $\kappa > 0$  such that for every line  $L$  passing through  $z \in V_{1/2} \cap U$  there exists  $\delta_{z,L} \in (\kappa, 1/2)$  such that  $C_{z,L} = \{\varphi > 0\} \cap B(z, \delta_{z,L}) \cap L$  is an  $\mathcal{F}$ -connected  $\mathcal{F}$ -neighborhood of  $z$  in  $L \cap V$ . Because  $z \in U$ ,  $C_{z,L}$  meets  $U$  in an  $\mathcal{F}$ -open subset of  $L$ . Therefore  $f \equiv -\infty$  on  $C_{z,L}$  according to Theorem 12.9 in [12]. It follows that  $f \equiv -\infty$  on the  $\mathcal{F}$ -open set

$$V \cap B(z, \kappa) \subset \bigcup_{L \ni z} C_{z,L}.$$

Now if  $|z| < \kappa$ , then  $0 \in V \cap B(z, \kappa)$ . The conclusion is that the  $\mathcal{F}$ -boundary of  $U$  does not hit  $\Omega$ , and therefore  $U = \Omega$ .  $\square$

Since pluripolar sets are  $\mathcal{F}$ -closed, Theorem 1.1 is a particular case of the following more general result.

**Corollary 5.3.** *Let  $U$  be an  $\mathcal{F}$ -domain in  $\mathbb{C}^n$ , and let  $E \subset \{f = -\infty\}$ , where  $f$  is  $\mathcal{F}$ -plurisubharmonic on  $U$  ( $\not\equiv -\infty$ ). Suppose that  $E$  is  $\mathcal{F}$ -closed. Then  $U \setminus E$  is an  $\mathcal{F}$ -domain.*

*Proof.* Suppose that  $U \setminus E = V \cup W$ , where  $V$  and  $W$  are non empty disjoint  $\mathcal{F}$ -open sets. Define  $h : U \setminus E \rightarrow [-\infty, \infty[$  by

$$(5.3) \quad h(z) = \begin{cases} 0 & \text{if } z \in V, \\ -\infty & \text{if } z \in W, \end{cases}$$

and define

$$(5.4) \quad \tilde{f}(z) = \begin{cases} f + h & \text{if } z \in V \cup W, \\ -\infty & \text{if } z \in E. \end{cases}$$

Then  $\tilde{f}$  is  $\mathcal{F}$ -upper semi-continuous. If we restrict  $\tilde{f}$  to a complex line, it is finely subharmonic. Indeed, on  $V$  because  $V \cap L$  is  $\mathcal{F}$ -open and  $\tilde{f} = f \not\equiv -\infty$ , and in  $U \setminus V$  there is nothing to prove because there  $\tilde{f} = -\infty$ . By Theorem 5.2,  $\tilde{f} \equiv -\infty$ , a contradiction  $\square$

One more consequence of Theorem 5.2 is the following maximum principle for  $\mathcal{F}$ -plurisubharmonic functions.

**Theorem 5.4.** *Let  $f \leq 0$  be  $\mathcal{F}$ -plurisubharmonic function on an  $\mathcal{F}$ -domain  $U$  in  $\mathbb{C}^n$ . Then either  $f < 0$  or  $f \equiv 0$ .*

*Proof.* Suppose that the  $\mathcal{F}$ -open set  $V = \{z \in U : f(z) < 0\}$  is not empty. The function  $g_n = nf$  is  $\mathcal{F}$ -plurisubharmonic. Since  $g_n$  decreases on  $V$ , the limit function  $g$  is  $\mathcal{F}$ -plurisubharmonic. By Theorem 5.2,  $g \equiv -\infty$  since it equals  $-\infty$  in  $V$ . Hence  $f < 0$ .  $\square$

## 6. APPLICATION TO PLURIPOLAR HULLS

In this final section we give a definition of  $\mathcal{F}$ -holomorphic functions of several complex variables. Next we prove a higher dimensional analog of Theorem 1.2 in [7]. See also Theorem 1 in [8].

**Definition 6.1.** Let  $U \subseteq \mathbb{C}^n$  be  $\mathcal{F}$ -open. A function  $f : U \rightarrow \mathbb{C}$  is said to be  $\mathcal{F}$ -holomorphic if every point of  $U$  has a compact  $\mathcal{F}$ -neighborhood  $K \subseteq U$  such that the restriction  $f|_K$  belongs to  $H(K)$ .

Here  $H(K)$  denotes the uniform closure on  $K$  of the algebra of holomorphic functions in a neighborhood of  $K$ .

**Lemma 6.2.** *Let  $U \subseteq \mathbb{C}^n$  be an  $\mathcal{F}$ -domain, and let  $f : U \longrightarrow \mathbb{C}$  be an  $\mathcal{F}$ -holomorphic function. Suppose that  $h : \mathbb{C}^2 \longrightarrow [-\infty, +\infty[$  is a plurisubharmonic function. Then the function  $z \mapsto h(z, f(z))$  is  $\mathcal{F}$ -plurisubharmonic on  $U$ .*

*Proof.* First, we assume that  $h$  is continuous and finite everywhere. Let  $a \in U$ . By Definition 6.1 there is a compact  $\mathcal{F}$ -neighborhood  $K$  of  $a$  and a sequence  $f_n$  of holomorphic functions defined in usual neighborhoods of  $K$  that converges uniformly to  $f|_K$ . Since  $h(z, f_n(z))$  is plurisubharmonic and converges uniformly to  $h(z, f(z))$  on  $K$ ,  $h(z, f(z))$  is  $\mathcal{F}$ -plurisubharmonic in the  $\mathcal{F}$ -interior of  $K$ .

Suppose that  $h$  is arbitrary. Then  $h$  is the limit of some decreasing sequence of continuous plurisubharmonic functions  $h_n \in \text{PSH}(\mathbb{C}^2)$ . By the first part of the proof,  $(h_n(z, f(z)))_n$  is a decreasing sequence of  $\mathcal{F}$ -plurisubharmonic functions in the  $\mathcal{F}$ -interior of  $K$ . The limit  $h(z, f(z))$  is therefore  $\mathcal{F}$ -plurisubharmonic in the  $\mathcal{F}$ -interior of  $K$ . Now by the sheaf property of  $\mathcal{F}$ -plurisubharmonic function we conclude that  $h(z, f(z))$  is  $\mathcal{F}$ -plurisubharmonic on  $U$ .  $\square$

A version of this lemma for functions of one variable with similar proof appears in [7].

The pluripolar hull  $E_\Omega^*$  of a pluripolar set  $E$  relative to an open set  $\Omega$  is defined as follows.

$$E_\Omega^* = \bigcap \{z \in \Omega : u(z) = -\infty\},$$

where the intersection is taken over all plurisubharmonic functions defined in  $\Omega$  which are equal to  $-\infty$  on  $E$ .

**Theorem 6.3.** *Let  $U \subset \mathbb{C}^n$  be an  $\mathcal{F}$ -domain. Let  $f$  be  $\mathcal{F}$ -holomorphic in  $U$ . Suppose that for some  $\mathcal{F}$ -open subset  $V \subset U$  the graph  $\Gamma_f(V)$  of  $f$  over  $V$  is pluripolar in  $\mathbb{C}^{n+1}$ . Then the graph  $\Gamma_f(U)$  of  $f$  is pluripolar in  $\mathbb{C}^{n+1}$ . Moreover,  $\Gamma_f(U) \subset (\Gamma_f(V))^*_{\mathbb{C}^{n+1}}$ .*

*Proof.* By Josefson's theorem, cf. [18], there exists  $h \in \text{PSH}(\mathbb{C}^{n+1})$  ( $\not\equiv -\infty$ ) such that  $h(z, f(z)) = -\infty$ ,  $\forall z \in V$ . In view of Lemma 6.2 and Theorem 5.2, the function  $h(z, f(z))$  is identically  $-\infty$  in  $U$ . It follows at once that  $\Gamma_f(U)$  is pluripolar and  $\Gamma_f(U) \subset (\Gamma_f(V))^*_{\mathbb{C}^{n+1}}$ .  $\square$

As a corollary we obtain a generalization to several complex variables of the main result of [8]. We keep the notation of Theorem 6.3.

**Corollary 6.4.** *Suppose that  $U$  contains a ball  $B$ . Then  $\Gamma_f(U)$  is pluripolar and  $\Gamma_f(U) \subset (\Gamma_f(B))^*_{\mathbb{C}^{n+1}}$ .*

*Proof.* On the intersection of  $B$  with any complex line  $f$  is a  $\mathcal{F}$ -holomorphic function of one variable, hence, holomorphic there, cf. [14], page 63. Thus  $f$  is holomorphic on  $B$  and  $\Gamma_f(B)$  is pluripolar. Now Theorem 6.3 applies.  $\square$

*Remark 6.5.* Theorem 6.3 and Corollary 6.4 only explain for a small part the propagation of pluripolar hulls. E.g., in the case of Corollary 6.4 take  $B$  the unit ball and consider the function  $g(z) = f(z)(z_1 - z_2^2)$ . Then, whatever the extendibility properties of  $f$  may be, the pluripolar hull of graph of  $g$  will contain the set  $\{z_1 = z_2^2\}$ .

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